

Tunability of the Fractional Quantum Hall States in Buckled Dirac Materials

Vadym M. Apalkov

Department of Physics and Astronomy, Georgia State University, Atlanta, Georgia 30303, USA

Tapash Chakraborty[†]

Department of Physics and Astronomy, University of Manitoba, Winnipeg, Canada R3T 2N2

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We report on the fractional quantum Hall states of germanene and silicene where one expects a strong spin-orbit interaction. This interaction causes an enhancement of the electron-electron interaction strength in one of the Landau levels corresponding to the valence band of the system. This enhancement manifests itself as an increase of the fractional quantum Hall effect gaps compared to that in graphene and is due to the spin-orbit induced coupling of the Landau levels of the conduction and valence bands, which modifies the corresponding wave functions and the interaction within a single level. Due to the buckled structure, a perpendicular electric field lifts the valley degeneracy and strongly modifies the interaction effects within a single Landau level: in one valley the perpendicular electric field enhances the interaction strength in the conduction band Landau level, while in another valley, the electric field strongly suppresses the interaction effects.

The unique electronic properties of graphene [2, 3], for example, that of the interacting Dirac fermions in an external magnetic field [4–7] have captivated our collective attention now for almost a decade. However, very recently, other emergent Dirac materials, such as silicene and germanene have rapidly gained considerable attention [8] because of their even greater promises. These systems are similar in structure as that of graphene, and hence contain those remarkable properties of graphene, and then some [9–11]. They are monolayers of silicon and germanium with hexagonal lattice structures where the low energy charge carriers are also massless Dirac fermions [12]. The interesting additional behavior lies in the *buckled* structure [13] of these systems due to their larger ionic radius than that of carbon, whereby the two sublattices in these systems are displaced vertically. As a consequence, a large spin-orbit interaction (SOI) induced gap opens up at the Dirac points ($\Delta_{\text{so}} \approx 1.55 - 7.9$ meV for silicene [10] and $\Delta_{\text{so}} \approx 24 - 93$ meV for germanene [10]). This is in contrast to the tiny SO gap of about 25 μeV in graphene [14]. The buckled structure of the lattice allows for the band gap to be tunable [15]. It has been suggested that with an applied perpendicular electric field the band gap can actually be controlled [16] as the size of the band gap increases linearly with the electric field strength. Quite naturally, this has generated a huge surge in interest in exploring the properties of Dirac fermions in these two systems, with an eye to their great potential for device applications. The fractional quantum Hall effect (FQHE) states of interacting Dirac fermions [4–7] are particularly intriguing in this context. The SOI is expected to significantly enhance the FQHE gap [17]. The large SO coupling in the present systems makes the FQHE states uniquely susceptible to an external control, and consequently a greater insight into the effect.

The FQHE in graphene has revealed some novel features specific to the relativistic systems. The electron-electron interactions are the strongest not in the $n = 0$ Landau level (LL) as in conventional semiconductor sys-

tems, but in the $n = 1$ LL, which results in the largest FQHE gaps in the $n = 1$ LL [4]. Here n is the LL index. The wave functions in the $n = 0$ LL in graphene are completely identical to the wave functions of the $n = 0$ LL of a conventional (nonrelativistic) system. The buckled structure of silicene and germanene not only modify their energy spectrum from that of graphene but the strong SOI also lifts the spin degeneracy, while an external electric field lifts the valley degeneracy of the energy levels. In a magnetic field, the LL spectra and the corresponding wave functions can be also modified and controlled by the electric field. Here we study the effects of the electric field on the correlation properties of the Dirac fermions in the FQHE regime. The measure of the strength of the electron-electron interactions can be characterized by the magnitude of the corresponding FQHE gaps. For bilayer graphene, it was shown [5, 6] that the bias voltage can strongly modify the property of the FQHE states and in some cases even increase the corresponding gap compared to that of the monolayer graphene. These effects are expected for monolayer silicene and germanene solely due to the SOI.

The low-energy Hamiltonian of the silicene/germanene monolayer is given by [16]

$$\mathcal{H}_\eta = v_F (p_x \tau_x - \eta p_y \tau_y) + \eta \tau_z h + L_z E_z \tau_z, \quad (1)$$

where

$$h = -\lambda_{\text{SO}} \sigma_z - a \hbar^{-1} \lambda_R (p_y \sigma_x - p_x \sigma_y), \quad (2)$$

$\eta = +1$ (K valley) and -1 (K' valley), τ_α and σ_α are the Pauli matrices corresponding to the sublattices (A and B) and the spin degrees of freedom, respectively. The parameters in Eqs. (1)-(2) are: v_F is the Fermi velocity, a is the lattice constant, λ_{SO} is the SO coupling, and λ_R is the intrinsic Rashba SO coupling. For germanene and silicene these parameters are $a = 4.063 \text{ \AA}$, $v_F = 7.26 \times 10^5 \text{ m/s}$, $L_z = 0.33 \text{ \AA}$, $\lambda_{\text{SO}} = 43 \text{ meV}$, $\lambda_R = 10.7 \text{ meV}$ for germanene and $a = 3.866 \text{ \AA}$, $v_F = 8.47 \times 10^5 \text{ m/s}$,

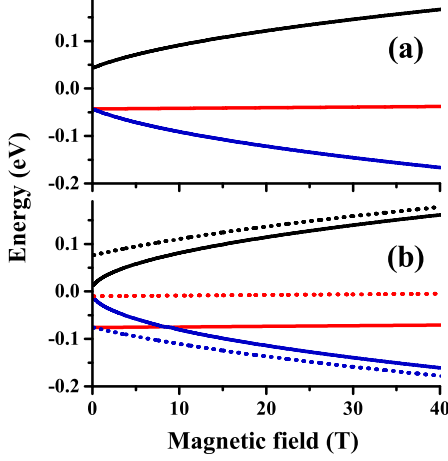


FIG. 1: Energies of three LLs corresponding to the Hamiltonian (3) as a function of the perpendicular magnetic field. The perpendicular electric field is (a) $E_z = 0$ and (b) $E_z = 0.1$ V/Å. For $E_z = 0$, each level has twofold valley degeneracy. The degeneracy is lifted for a finite electric field [panel (b)], where the levels of the K valley are shown by solid lines, and the levels of K' valley are shown by dashed lines. The results shown here are for germanene.

$L_z = 0.23$ Å, $\lambda_{SO} = 3.9$ meV, $\lambda_R = 0.7$ meV for silicene. The wave functions corresponding to the Hamiltonian (1) have four components of the form $(\psi_{A\uparrow}, \psi_{B\uparrow}, \psi_{A\downarrow}, \psi_{B\downarrow})$, where $\psi_{A\alpha}$ and $\psi_{B\alpha}$ determine the amplitude of the wave function in sublattice A and B, respectively, with spin direction $s = \uparrow, \downarrow$. In a magnetic field, the momentum \vec{p} is replaced by the generalized momentum $\vec{\pi} = \vec{p} + e\vec{A}/c$, where \vec{A} is the vector potential. To describe the LL wave functions, it is convenient to introduce the Landau functions of the nonrelativistic system $\phi_{n\uparrow}(\phi_{n\downarrow})$ in the corresponding nonrelativistic LL with index n and spin direction \uparrow or \downarrow . Then the structure of the LL wave functions in silicene/germanene can be schematically (without the coefficients) described as $(\phi_{n\uparrow}, \phi_{n+1\uparrow}, \phi_{n-1\downarrow}, \phi_{n\downarrow})$ (K valley) and $(\phi_{n+1\uparrow}, \phi_{n\uparrow}, \phi_{n\downarrow}, \phi_{n-1\downarrow})$ (K' valley).

The FQHE is expected only in those LLs whose wave functions are mixtures of ϕ_0 and ϕ_1 [6, 7]. There are two types of such LLs in silicene/germanene. The first type has the wave function of the form $(0, \phi_{0\uparrow}, 0, 0)$ (for K valley). This LL consists of only ϕ_0 and the interaction in this LL is exactly the same as that in the $n = 0$ nonrelativistic semiconductor system or in graphene. The corresponding gaps are exactly the same as in a nonrelativistic system and they do not depend on the external electric field.

The second type of the wave functions has the form $\Psi_2 = (C_1\phi_{0\uparrow}, C_2\phi_{1\uparrow}, 0, C_3\phi_{0\downarrow})$ (we consider only the K valley and include the coefficients C_1, C_2 , and C_3 in the wave functions). The wave functions Ψ_2 effectively have three components and they are mixtures of the $n = 0$ and $n = 1$ nonrelativistic functions. In the basis of functions

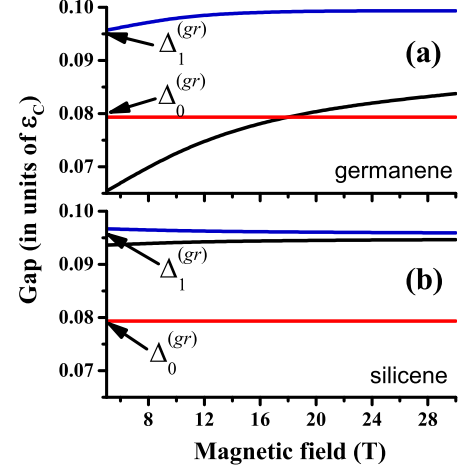


FIG. 2: The $\nu = 1/3$ gap in three LLs corresponding to the Hamiltonian (3) as a function of the magnetic field. The electric field is zero. The results are shown for (a) germanene and (b) silicene. The color of the lines corresponds to the color of the LL shown in Fig. 1(a). The $1/3$ gaps of the graphene monolayer in the $n = 0$ ($\Delta_0^{(gr)}$) and $n = 1$ ($\Delta_1^{(gr)}$) LLs are also shown. The finite size system has eight electrons.

Ψ_2 the Hamiltonian has the 3×3 matrix form

$$\mathcal{H}_2 = \begin{pmatrix} -\lambda_{SO} + S_+ & \hbar\omega_B & 0 \\ \hbar\omega_B & \lambda_{SO} - S_- & -i\sqrt{2}(a/\ell_0)\lambda_R \\ 0 & i\sqrt{2}(a/\ell_0)\lambda_R & -\lambda_{SO} - S_+ \end{pmatrix} \quad (3)$$

where $\ell_0 = (\hbar/eB)^{1/2}$ is the magnetic length, $\omega_B = \sqrt{2}v_F/\ell_0$, and we have introduced the notations $S_+ = L_z E_z + \Delta_z$, $S_- = L_z E_z - \Delta_z$. Here $\Delta_z = g\mu_B B$ is the Zeeman energy and we assumed that the g -factor in germanene/silicene is close to that of graphene $g \approx 2.2$. The eigenvalues and eigenvectors of the matrix Hamiltonian (3) determine the energies of three LLs and the corresponding wave functions (coefficients C_1, C_2, C_3).

In the FQHE regime a given LL is partially occupied and the ground state of the electron system and the corresponding excitations are completely determined by the electron-electron interactions. The strength of the electron-electron interactions can be described by the Haldane's pseudopotentials, V_m , [19] which are the energies of two electrons with relative angular momentum m . For the wave function $\Psi_2 = (C_1\phi_{0\uparrow}, C_2\phi_{1\uparrow}, 0, C_3\phi_{0\downarrow})$, which is characterized by the coefficients C_1, C_2 , and C_3 , the Haldane's pseudopotentials are

$$V_m = \int_0^\infty \frac{dq}{2\pi} q V(q) [F(q)]^2 L_m(q^2) e^{-q^2}, \quad (4)$$

where $L_m(x)$ are the Laguerre polynomials, $V(q) = 2\pi e^2/(\kappa\ell_0 q)$ is the Coulomb interaction in the momentum space, κ is the dielectric constant, and $F(q)$ is the corresponding form factor,

$$F(q) = (|C_1|^2 + |C_3|^2) L_0(q^2/2) + |C_2|^2 L_1(q^2/2), \quad (5)$$

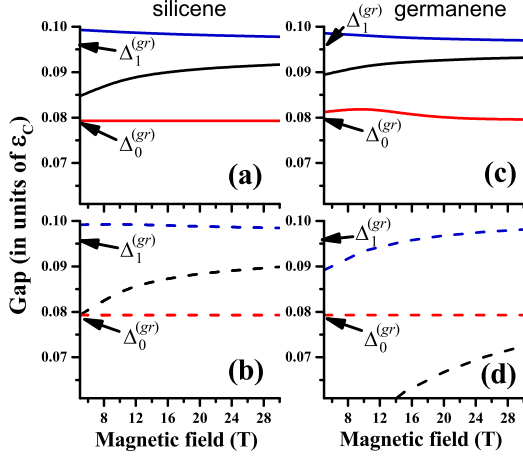


FIG. 3: The $\nu = 1/3$ gap for three LLs corresponding to the Hamiltonian (3) as a function of the magnetic field. The electric field is $E_z = 0.1 \text{ V/\AA}$. The results are for (a,b) silicene and (c,d) germanene. The color of the lines corresponds to the color of the LL in Fig. 1. The panels (a,c) correspond to the K valley, while the panels (b,d) correspond to the K' valley. The $1/3$ gaps of the graphene monolayer in the $n = 0$ ($\Delta_0^{(gr)}$) and $n = 1$ ($\Delta_1^{(gr)}$) LLs are also shown.

To explore the correlation effects and the strength of electron-electron interactions in a many electron germanene/silicene system we consider below the partially occupied LL with a fractional filling factor corresponding to the FQHE [18]. We study the many-electron system at fractional filling factors numerically within the spherical geometry [19]. With the known Haldane's pseudopotentials (4) we determine the interaction Hamiltonian matrix [20] and then numerically evaluate a few lowest eigenvalues and eigenvectors of this matrix. The FQHE is observed when the ground state of the system is an incompressible liquid, the energy spectrum of which has a finite many-body gap. The magnitude of the gap indicates the interaction strength within a single LL and also determines the stability of the FQHE state. In conventional nonrelativistic system the FQHE is observed only in two lowest Landau levels, while in the higher Landau levels the charge density wave with gapless excitations has lower energy [21]. The most stable FQHE state, i.e., with the largest gap, is realized in the $n = 0$ LL in conventional nonrelativistic systems and in the $n = 1$ LL in a graphene monolayer.

In Fig. 1 the energy spectra, corresponding to the Hamiltonian (3) and consisting of three LLs, is shown for germanene as a function of the magnetic field. The results are for zero electric field [Fig. 1(a)] and for $E_z = 0.1 \text{ V/\AA}$ [Fig. 1(b)]. For $E_z = 0$, each LL has twofold valley degeneracy, which is lifted for a finite electric field. In Fig. 1(b) the LL of the K and K' valleys are shown by solid and dashed lines, respectively. Without the SOI and for zero electric field, three LLs correspond to three LLs of graphene with energies $\varepsilon = 0$ ($n = 0$ LL), and

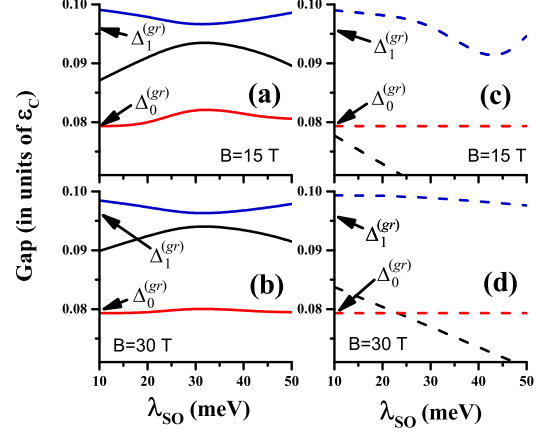


FIG. 4: The $\nu = 1/3$ gap in three LLs corresponding to the Hamiltonian (3) as a function of the SOI (λ_{SO}). The electric field is $E_z = 0.1 \text{ V/\AA}$. The results are for germanene. The color of the lines corresponds to the color of the LL shown in Fig. 1. The panels (a,b) correspond to K valley, while the panels (c,d) correspond to K' valley. The magnetic field is (a,c) $B = 15 \text{ T}$ and (b,d) $B = 30 \text{ T}$. The $1/3$ -FQHE gaps of the graphene monolayer in the $n = 0$ ($\Delta_0^{(gr)}$) and the $n = 1$ ($\Delta_1^{(gr)}$) LLs are also shown.

$\varepsilon = \pm \hbar \omega_B \propto \pm \sqrt{B}$ ($n = \pm 1$ LLs). The SOI couples these states, which results in mixing of the corresponding wave functions and shifting of the energy levels, which is clearly seen in Fig. 1. The level shown by a red line in Fig. 1, and which originated from the $n = 0$ LL of graphene, has a weak magnetic field dependence. For zero electric field the energy spectra can be found analytically and the three LLs shown in Fig. 1(a) have energies $\varepsilon = \lambda_{SO}$ [red line in Fig. 1(a)] and $\varepsilon = \pm \sqrt{\lambda_{SO}^2 + \hbar^2 \omega_B^2 (1 + a^2 \lambda_R^2 / \hbar^2 v_F^2)}$ (black and blue lines). In this case the energy and the structure of the LL, shown by red line in Fig. 1(a), do not depend on the magnetic field. The wave functions of this LL consist of ϕ_0 -type of nonrelativistic wave functions only. In a finite perpendicular electric field E_z [see Fig. 1(b)], the LLs, shown by red lines in Fig. 1(b), acquire a weak magnetic field dependence. Strong lifting of the valley degeneracy is also observed in Fig. 1(b). The data in Fig. 1 are for germanene. For silicene, the results are similar but with a smaller energy scale due to the weaker SOI in silicene. It is convenient to label the LLs shown in Fig. 1 following the labeling scheme of graphene: $n = -1$, $n = 0$, and $n = 1$ LLs are shown by blue, red, and black lines, respectively.

The $\nu = 1/3$ gaps for different LLs are shown in Fig. 2 for zero electric field. The gaps $\Delta_0^{(gr)}$ and $\Delta_1^{(gr)}$ of graphene in the $n = 0$ and the $n = 1$ LLs are also shown. For the $n = 0$ germanene/silicene LL, the gap is exactly the same as that of graphene ($\Delta_0^{(gr)}$). The difference between the gaps in the $n = -1$ and $n = 1$ LLs of germanene/silicene and the gap $\Delta_1^{(gr)}$ of graphene illustrates

the SO-induced coupling of the $n = -1$ and $n = 1$ LLs. For silicene, the SO coupling is weak, which results in a small deviation of the gaps from the $\Delta_1^{(\text{gr})}$ value and a small splitting of the $n = -1$ and $n = 1$ gaps. The strong SO coupling in germanene results in a large splitting of the gaps in the $n = -1$ and $n = 1$ LLs [see Fig. 2(a)]. The difference between the gaps in these LLs becomes smaller with increasing magnetic field. The FQHE in the $n = -1$ LL [blue line in Fig. 2(a)] is always greater than the largest gap $\Delta_1^{(\text{gr})}$ in graphene, while the gap in the $n = 1$ LL [black line in Fig. 2(a)] is strongly suppressed, especially for small magnetic fields. Due to twofold valley degeneracy of the LLs in zero electric field, the behavior shown in Fig. 2 is the same for both valleys, K and K' .

In a finite electric field, the LLs in different valleys have different properties, which results in different values of the gaps. In Fig. 3 the gaps in the LLs of germanene and silicene are shown as a function of the magnetic field for $E = 0.1 \text{ V/\AA}$ and different valleys. While the gap in the $n = 0$ LL, similar to the case of $E_z = 0$, has a weak magnetic field dependence, the gaps in the $n = 1$ and $n = -1$ LLs are strongly modified compared to the $E_z = 0$ case. The changes in the behavior of the gaps are however quite different for silicene and germanene.

For silicene [see Fig. 3(a,b)], the application of a perpendicular electric field strongly increases the difference between the values of the gaps in the $n = 1$ and $n = -1$ LLs. The gap in the $n = -1$ LL is larger, while the gap in the $n = 1$ LL is smaller than the gap $\Delta_1^{(\text{gr})}$ in graphene. The behavior of the gaps is the same for both valleys, while for the K' valley the gap in the $n = 1$ LL is smaller than the one in the K valley. A different situation occurs for germanene, where the electric field reduces the difference between the gaps in the $n = -1$ and $n = 1$ LLs for the K valley and increase this difference for the K' valley [Fig. 3(c,d)]. The electric field also suppresses the gap in the $n = -1$ LL for the K' valley [Fig. 3(d)]. For small magnetic fields, $B \lesssim 15 \text{ T}$, the gap in the $n = -1$ LL of the K' valley becomes even less than the gap $\Delta_1^{(\text{gr})}$. This

behavior shows a strong sensitivity of the gaps on the magnitudes of the applied electric field and the magnetic field.

The above results illustrate the importance of the SOI in determining the properties of the graphene-like systems. To illustrate the effect of the SOI on the value of the gaps, we vary the SO parameters, λ_{SO} , keeping all other parameters constant and equal to the parameters of germanene. The corresponding dependence of the gaps on λ_{SO} is shown in Fig. 4 for an electric field $E_z = 0.1 \text{ V/\AA}$ and for $B = 15 \text{ T}$ and $B = 30 \text{ T}$. The gaps show clear nonmonotonic dependence on λ_{SO} with a local maximum (minimum) at $\lambda_{\text{SO}} \approx 35 \text{ meV}$. For the K valley, both for the $n = -1$ and $n = 1$ LLs the gaps are large and comparable to that in graphene. For K' the behavior is different. While for the $n = -1$ LL the gap is large and has a minimum at $\lambda_{\text{SO}} \approx 45 \text{ meV}$ (at $B = 15 \text{ T}$), the gap for the $n = 1$ LL is strongly suppressed with increasing λ_{SO} . The suppression is stronger for smaller magnetic fields.

In conclusion, we have shown that in graphene-like systems such as germanene and silicene, which have a strong SO interaction, there is an enhancement of the electron-electron interaction strength in one of the LL levels, which corresponds to the valence band of the system. This enhancement manifests itself as an increase of the gaps compared to that of graphene and is due to the SO-induced coupling of the LLs of the conduction and valence bands, which modifies the corresponding wave functions and the interaction within a single LL. A perpendicular electric field lifts the valley degeneracy of the systems and strongly modifies the interaction effects within a single LL. In one valley the electric field enhances the interaction strength (and the corresponding gaps) in the conduction band LL, while in another valley, the electric field strongly suppresses the interaction effects.

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- [†] Electronic address: Tapash.Chakraborty@umanitoba.ca
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